

Can one increase the luminosity of a Schwarzschild black hole?

Avraham E. Mayo*

*The Racah Institute of Physics, Hebrew University of Jerusalem,
Givat Ram, Jerusalem 91904, Israel*

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We illustrate how Hawking's radiance from a Schwarzschild black hole is *modified* by the electrostatic self-interaction of the emitted charged particles. A W.K.B approximation shows that the probability for a self-interacting charged particle to propagate from the interior to the exterior of the horizon is increased relative to the corresponding probability for neutral particles. We also demonstrate how the electric potential of a charged test object in the black hole's vicinity gives rise to pair creation. We analyze this phenomenon semiclassically by considering the existence of the appropriate *Klein region*. Finally we discuss the possible energy source for the process.

I. INTRODUCTION

The aim of this work is to study various effects that may take place when a charged object propagates on a background Schwarzschild spacetime. As elucidated by Vilenkin [1] and corroborated by Smith and Will [2] and Lohiya [3], in contrast to the situation in flat spacetime, in the presence of a Schwarzschild black hole the *self-field* of the object, makes a nontrivial contribution to the object's energy as measured at infinity. Alternatively, this phenomenon may be interpreted as black hole polarization; the test object polarize the black hole so that image charges are formed on the black hole horizon. It was shown by various authors [4] that this effect is of unquestionable importance in black hole thermodynamics. One may think of two additional *distinct* effects that may take place on account of black hole polarization. The first is the emission of self-interacting charged particles. Here the black hole is *isolated*. The temperature of the black hole is assumed to be high enough so that the lightest massive charged particles may be emitted. The second effect is pair creation due to vacuum polarization. The black hole is allowed to interact with an *exterior* charged test object, and the potential generated by the charge is taken into account. Both processes seems to have the effect of increasing the luminosity of the black hole in a sense that will be clarified below.

In the following sections we investigate these two effects. First we show that once the temperature of the Schwarzschild black hole $\sim 1/M$, is equal or greater than the rest-mass μ of a self-interacting charged particle ($\mu \lesssim 1/M$), the emission of these particles is more probable than the emission of similar neutral particles by a factor $\exp(\pi e^2)$, where e is the charge (unless the contrary is stated we use "natural units" in which $\hbar = c = G = k = 1$

*Electronic mail: Mayo@cc.huji.ac.il

throughout). This is a direct result of the self-interaction of the charged particles. In analogy with the increased thermionic emission from a hot filament placed in a repelling electrostatic field, which has the effect of reducing the potential barrier at the edge of the metal, the effective electrostatic self-field experienced by the charges has the effect of increasing the emission [5]. In order to demonstrate this we compute in Sec.II the probabilities for finding particles in the exterior and interior of the horizon using the appropriate wave equation coupled to an effective electromagnetic self-field. It turns out that for a Schwarzschild black hole, the chemical potential is lowered (for both signs of charge) by $e^2/8M$. As pointed out before by several researches [1,3], this suggests that the emission of charged particles would increase relative to the emission of neutral particles. We calculate an approximation to the absorption coefficient (“greybody factor”) for the problem. We find that when the energy of the emitted particles is close to $e^2/8M$, the absorption coefficient is of the form $a(\omega, \mu, e^2/8M)(\omega - e^2/8M)$ plus small corrections. This means that in the energy range $\mu < \omega < e^2/8M$, the black hole should start to superradiate.

Thus far we have discussed the emission of charged self-interacting particles. In Sec.III we change the settings and discuss the consequences of placing a charged test object in the *exterior* of a Schwarzschild black-hole. We illustrate how the field generated by the charged particle gives rise to pair production. We analyze the problem semiclassically and find the energy range for the existence of a Klein region. It was previously thought that this energy range does not exist in the case of an *isolated* nonrotating neutral black-hole [6].

In Sec. IV we address the problem of possible energy sources for the process. Thermodynamical arguments suggest that if the black-hole is too massive, the entropy carried away by the pairs is not sufficient to compensate for the entropy lost by the black-hole. Accordingly we look into the dynamics of the test object, and investigate the role it may play in the supply of energy for the process. Finally in Sec. V we summarize our findings.

II. PARTICLE EMISSION RATE.

We take now a semiclassical approach to the calculation of the transmissions probabilities. We will use the complex path method which was recently advanced by various authors [7–9]. Specifically Srinivasan and Padmanabhan [7] proposed a derivation of Hawking radiation without using the Kruskal extension as in [10]. The point, of course, is that the Schwarzschild coordinates possess a coordinate singularity at the horizon. This bad behavior of the coordinates manifests itself as singularity in the expression for the semiclassical propagator near the horizon and a specific prescription to bypass it must be provided. The action functional may be constructed using the Hamilton-Jacobi method in the appropriate coordinates. It will be shown how the analytic continuation used in complex path method gives the result that the probability for particles to be found in the exterior of the horizon is not the same as the probability for particles to be found in the interior of the horizon. The ratio between these probabilities is of the form, $P_{\text{interior}} = \chi \exp(-\beta\varepsilon) P_{\text{exterior}}$, where ε is the energy of the particles and $\beta = 8\pi M$ is the inverse Hawking temperature. χ is a factor of the form $\chi = \exp(\pi e^2)$, where e is the charge of the test particle. This implies that the usual thermal distribution of neutral particles is modified by the addition of what may be interpreted as a charge dependent chemical potential.

A. Transmission probabilities

In a Schwarzschild spacetime consider a minimally coupled test scalar field Φ with mass μ coupled to the electromagnetic self-field A_α generated by the charge e . The scalar field propagates in the metric

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\Omega^2, \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, and satisfies

$$\left(\nabla_\alpha - \frac{i}{\hbar}eA_\alpha\right)\left(\nabla^\alpha - \frac{i}{\hbar}eA^\alpha\right)\Phi = \frac{\mu^2}{\hbar^2}\Phi. \quad (2.2)$$

A_α , linear in e , is the self-potential whose source is the object itself [1–3].

$$A_\alpha = -\frac{eM}{2r^2}\delta_\alpha^t. \quad (2.3)$$

A_α is divergence free accurate to $O(e^2)$. Using this and expanding the left hand side of Eq. (2.2), we obtain,

$$\frac{1}{r^2}\partial_r[r^2(1 - 2M/r)\partial_r\Phi] - (1 - 2M/r)^{-1}\left(\partial_t - \frac{i}{\hbar}eA_t\right)^2\Phi - \frac{1}{r^2}\hat{L}^2\Phi = \frac{\mu^2}{\hbar^2}\Phi, \quad (2.4)$$

where \hat{L}^2 is the usual squared angular momentum operator with eigenvalues $L^2 = \hbar^2 l(l+1)$.

Since the problem is a spherically symmetric one, we put $\Phi = \exp((i/\hbar)\mathcal{S}(r, t))Y_l^m(\theta, \varphi)$ where \mathcal{S} is a function which will be expanded in powers of \hbar . Substituting in the above equation, we obtain,

$$\begin{aligned} &[-(1 - 2M/r)^{-1}(\partial_t\mathcal{S} - eA_t)^2 + (1 - 2M/r)(\partial_r\mathcal{S})^2 + (\mu^2 + L^2/r^2)] \\ &+ \frac{\hbar}{i}\left[-(1 - 2M/r)^{-1}\partial_t^2\mathcal{S} + \frac{1}{r^2}\partial_r(r^2(1 - 2M/r))\partial_r\mathcal{S} + (1 - 2M/r)\partial_r^2\mathcal{S}\right] = 0. \end{aligned} \quad (2.5)$$

Expanding $\mathcal{S}(r, t) = \mathcal{S}_0(r, t) + (\hbar/i)\mathcal{S}_1(r, t) + (\hbar/i)^2\mathcal{S}_2(r, t) + \dots$, substituting into Eq. (2.5) and neglecting terms of order \hbar/i and higher, we find to lowest order,

$$-(1 - 2M/r)^{-1}(\partial_t\mathcal{S}_0 - eA_t)^2 + (1 - 2M/r)(\partial_r\mathcal{S}_0)^2 + (\mu^2 + L^2/r^2) = 0. \quad (2.6)$$

This is just the Hamilton-Jacobi equation satisfied by a charged particle of mass μ and charge e moving in the spacetime (2.1) and interacting with the potential A_α as given in Eq. (2.3). The solution to the above equation is

$$\mathcal{S}_0(r, t) = -\varepsilon t \pm \int^r dr (1 - 2M/r)^{-1} \sqrt{(\varepsilon + eA_t)^2 - (1 - 2M/r)(\mu^2 + L^2/r^2)}, \quad (2.7)$$

where ε is a constant identified with the energy.

The semiclassical kernel $\mathcal{K}(r_2, t_2; r_1, t_1)$ for the particle to propagate from (t_1, r_1) to (t_2, r_2) in the saddle point approximation can be written down immediately:

$$\mathcal{K}(r_2, t_2; r_1, t_1) = \mathcal{N} \exp \left(\frac{i}{\hbar} \mathcal{S}(r_2, t_2; r_1, t_1) \right). \quad (2.8)$$

\mathcal{N} is a suitable normalization constant, and \mathcal{S} is the action functional satisfying the classical Hamilton-Jacobi equation, namely

$$\mathcal{S}(r_2, t_2; r_1, t_1) = \mathcal{S}_0(r_2, t_2) - \mathcal{S}_0(r_1, t_1). \quad (2.9)$$

The sign ambiguity in \mathcal{S}_0 is related to the “outgoing” ($\partial_r \mathcal{S} > 0$) or “ingoing” ($\partial_r \mathcal{S} < 0$) nature of the particle. As long as points 1 and 2, between which the transition amplitude is calculated, are on the same side of the horizon (i.e. either exterior or interior to the horizon), the integral in the action is well defined and real. But if the points are located on opposite sides of the horizon, then the integral does not exist due to the divergence of the integrand at $r = 2M$. Therefore, in order to obtain the probability amplitude for crossing the horizon we have to give a prescription for evaluating the integral. The prescription used by Srinivasan and Padmanabhan [7], and adopted here, is to take the contour defining the integral to be an infinitesimal semicircle *above* the pole at $r = 2M$ for outgoing particles on the left of the horizon and ingoing particles on the right. Similarly, for ingoing particles on the left and outgoing particles on the right of the horizon (which corresponds to a time reversed situation of the previous cases), the contour should be an infinitesimal semicircle *below* the pole at $r = 2M$.

Consider, therefore, an outgoing particle ($\partial_r \mathcal{S} > 0$) at $r = r_1 < 2M$. The squared modulus of the amplitude for this particle to cross the horizon gives the probability to find the particle in the exterior of the horizon. The contribution to \mathcal{S} in the ranges $(r_1, 2M - \delta)$ and $(2M + \delta, r_2)$ is real. Therefore, choosing the contour to lie in the upper complex plane,

$$\begin{aligned} \mathcal{S}_{r_1 \xrightarrow{2M} r_2} &= - \lim_{\delta \rightarrow 0} \int_{2M-\delta}^{2M+\delta} \frac{\varepsilon + eA_t}{1 - 2M/r} dr + (\text{real part}) \\ &= i2\pi M(\varepsilon + A_t(2M)) + (\text{real part}) \end{aligned} \quad (2.10)$$

where A_t is evaluated at $r = 2M$. The minus sign in front of the integral corresponds to the initial condition that $\partial_r \mathcal{S} > 0$ at $r = r_1 < 2M$. The same result is obtained when an ingoing particle ($\partial_r \mathcal{S} < 0$) is considered at $r = r_1 < 2M$. The contour for this case must be chosen to lie in the lower complex plane. The amplitude for this particle to cross the horizon is the same as that of the outgoing particle due to the time reversal symmetry obeyed by the system.

Consider next, an ingoing particle ($\partial_r \mathcal{S} < 0$) at $r = r_2 > 2M$. The squared modulus of the amplitude for this particle to cross the horizon gives the probability to find the particle in the interior of the horizon. Choosing the contour to lie in the upper complex plane, we get,

$$\begin{aligned} \mathcal{S}_{r_1 \xleftarrow{2M} r_2} &= - \lim_{\delta \rightarrow 0} \int_{2M+\delta}^{2M-\delta} \frac{\varepsilon + eA_t}{1 - 2M/r} dr + (\text{real part}) \\ &= -i2\pi M(\varepsilon + eA_t(2M)) + (\text{real part}) \end{aligned} \quad (2.11)$$

where as before A_t is evaluated at $r = 2M$. The same result is obtained when an outgoing particle ($\partial_r \mathcal{S} > 0$) is considered at $r = r_2 > 2M$. The contour for this case should be in the

lower complex plane and the amplitude for this particle to cross the horizon is the same as that of the ingoing particle due to time reversal invariance.

Taking the modulus square to obtain the probability, and substituting for A_t from Eq. (2.3), we get,

$$\left(\begin{array}{c} \text{probability for a particle} \\ \text{with energy } \varepsilon \\ \text{to be found at } r > 2M \end{array} \right) = \exp(-8\pi M\varepsilon + \pi e^2) \left(\begin{array}{c} \text{probability for a particle} \\ \text{with energy } \varepsilon \\ \text{to be found at } r < 2M \end{array} \right). \quad (2.12)$$

This suggest that it is more likely for a particular region to gain particles than lose them. Therefore we must interpret the above result as saying that the probability of emission of particles is not the same as the probability of absorption of particles. Furthermore, this result implies that the flux of particle at infinity in this case would be greater by a factor $\exp(\pi e^2)$ than the flux of neutral particles of the same mass and spin. However, it should be noted that the increase factor is in fact negligibly small when one considers the emission of electrons. In that case the numerical factor is equal to $\exp(\pi e^2/(\hbar c)) \sim \exp(\pi/137) \approx 1.023$. This result obviously cast doubt on the astrophysical importance of the electrostatic self-interaction.

Note that the method presented here for the calculation of the modulation factor of Hawking's radiance is not specific to the electrostatic self-potential. It would also apply for an *external* electrostatic potential (as opposed to the *internal* electrostatic self-potential). Provided that the external potential is analytic in the neighborhood of the horizon, the thermal radiation is modulated by varying the magnitude of the external potential. The modulation factor is of the form $\exp(eA_t^{ext}(2M)/T_H)$, where the external potential, A_t^{ext} is evaluated at the horizon and T_H is the Hawking temperature.

Recently van Putten [11] discussed the magnetic analog of this phenomenon. He showed that a rotating black-hole produces electron-positron outflow when immersed in a magnetic field. The outflow is driven primarily by a coupling of the spin of the black hole to the fermionic wave-function. The source for the magnetic field is assumed to be exterior. He computed, in the WKB approximation, the superradiant amplification of scalar waves confined to a thin equatorial wedge around a Kerr black-hole and found it to be higher than for radiation incident over all angles. However, Aguirre [12] has recently presented calculations of both spin-0 (scalar) superradiance (integrating the radial equation rather than using the WKB method) and spin-1 (electromagnetic/magnetosonic) superradiance in van Putten's wedge geometry, and showed that in contrast to the scalar case, spin-1 superradiance is weaker in the wedge geometry. So that, as with the electrostatic self-interaction case, the astrophysical significance of the effect is questionable.

B. The greybody factor

It follows from the previous subsection that Hawking emission of charged self-interacting particles from the black-hole gives rise to a flux at infinity

$$\frac{d^2 n}{d\omega dt} = \frac{1}{2\pi} \frac{\Gamma}{e^{8\pi M(\omega - e^2/8M)} - 1}, \quad (2.13)$$

where ω is the energy of the particle at infinity, e is the charge of the particles, $8\pi M$ is the inverse of Hawking temperature, and Γ is the relevant absorption factor, the so called “greybody factor”. From the equation above it is apparent that the term $-e^2/8M$ serves as a chemical potential in the problem; intriguingly, this chemical potential is the same for both particles and antiparticles (usually chemical potential has opposite sign for particles and antiparticles). It may be seen that this chemical potential enhances the emission of (positive or negative) charged particles over that of neutral particles with the same mass and spin. Positive and negative charged particles are treated equally.

It turns out that two different situations occur for $\omega > e^2/8M$ and $\omega < e^2/8M$. To see this, expand the right hand side of Eq. (2.13) around $\omega = e^2/8M$. Then, we get that the flux at infinity is

$$\frac{d^2 n}{d\omega dt} \approx \frac{1}{2\pi} \frac{\Gamma}{8\pi M(\omega - e^2/8M)}. \quad (2.14)$$

It is now evident that for $\omega > e^2/8M$ the flux is positive. However for $\omega < e^2/8M$ the flux seems to be negative, with a singularity at $\omega = e^2/8M$! None the less, one shouldn't be perplexed by this. It is just that we have failed to take into account the energy dependence of the absorption factor Γ . Obviously, we must show that the the point of transition occurs at a zero of Γ , namely the expansion of the absorption factor around $\omega = e^2/8M$ must begin with

$$\Gamma \approx a(\omega, \mu, e^2/8M)(\omega - e^2/8M) + \dots, \quad (2.15)$$

where $a(\omega, \mu, e^2/8M)$ is finite function of its variables.

Therefore, we turn to calculate Γ to first approximation. Making the ansatz $\Phi = e^{-i\omega t} Y_l^m(\theta, \varphi) f(r)/r$ and substituting into Eq. (2.4) we obtain an effective Schrödinger equation

$$\begin{aligned} -\hbar^2 \frac{d^2 f}{dr^{*2}} + V(r^*) f &= 0, \\ V(r^*) &= \left(1 - \frac{2M}{r}\right) \left(\mu^2 + \frac{L^2}{r^2} + \frac{\hbar^2 2M}{r^3}\right) \\ &\quad - \left(\omega - \frac{e^2 M}{2r^2}\right)^2. \end{aligned} \quad (2.16)$$

Here $r^* = r + 2M \log(r - 2M)$ is Wheeler's “tortoise” coordinate [13]. We are interested in asymptotic solutions of the equation in the *far region* $r^* \rightarrow \infty$ ($r \rightarrow \infty$) and in the *near region*, $r^* \rightarrow -\infty$ ($r \rightarrow 2M$).

Taking the limit of $V(r^*)$ in the *far region*, $r^* \rightarrow \infty$ ($r \rightarrow \infty$), we find that the equation has the form

$$-\hbar^2 \frac{d^2 f}{dr^{*2}} + (\mu^2 - \omega^2) f = 0. \quad (2.17)$$

For $\mu > \omega$ this equation has exponentially decaying and diverging solutions. Obviously, the diverging solution is unacceptable on physical grounds, while the decaying solution is associated with a particle trapped in the effective potential well. For $\mu < \omega$ Eq. (2.17) has

an ingoing and an outgoing wave solutions. Since we are actually dealing with a scattering problem, we concentrate on those solutions:

$$f(r \rightarrow \infty) = e^{-i\sqrt{\omega^2 - \mu^2}r^*/\hbar} + \mathcal{A}e^{i\sqrt{\omega^2 - \mu^2}r^*/\hbar}. \quad (2.18)$$

Here \mathcal{A} is a constant to be determined later.

In the *near region*, $r^* \rightarrow \infty$ ($r \rightarrow 2M$), we find that Eq. (2.16) has the limiting form

$$-\hbar^2 \frac{d^2 f}{dr^{*2}} - (\omega - e^2/8M)^2 f = 0. \quad (2.19)$$

The Matzner boundary condition [14] that the physical solution be an ingoing wave, as appropriate to the absorbing character of the horizon, selects the solution of the above equation to be

$$f(r \rightarrow 2M) = \mathcal{B}e^{-i(\omega - e^2/8M)r^*/\hbar}, \quad (2.20)$$

where \mathcal{B} is a constant.

\mathcal{A} and \mathcal{B} may be determined by matching f and f' of the solutions in the *far and near regions* at some point in the intermediate region $2M \ll r \ll \infty$. Doing so we find

$$\begin{aligned} \mathcal{A} &= \frac{\omega - e^2/8M - \sqrt{\omega^2 - \mu^2}}{\omega - e^2/8M + \sqrt{\omega^2 - \mu^2}} e^{-2\frac{i}{\hbar}\sqrt{\omega^2 - \mu^2}r_m^*}, \\ \mathcal{B} &= \frac{2\sqrt{\omega^2 - \mu^2}}{\omega - e^2/8M + \sqrt{\omega^2 - \mu^2}} e^{-\frac{i}{\hbar}(\sqrt{\omega^2 - \mu^2} - (\omega - e^2/8M))r_m^*}, \end{aligned} \quad (2.21)$$

where r_m^* is the matching point. The absorption cross-section may now be obtained using the method of fluxes. The flux in our one-dimensional effective problem is

$$\mathcal{F} = \frac{\hbar}{2i} (f^* \partial_r f - c.c.). \quad (2.22)$$

The absorption probability is the ratio of the incoming flux at the horizon to the incoming flux at infinity,

$$\begin{aligned} \Gamma &= \frac{\mathcal{F}_{\mathcal{H}}}{\mathcal{F}_{\infty}^{incoming}} = \\ &= \frac{4\sqrt{\omega^2 - \mu^2}}{(\omega - e^2/8M - \sqrt{\omega^2 - \mu^2})^2} (\omega - e^2/8M). \end{aligned} \quad (2.23)$$

The matching point r_m^* has disappeared from the expression for Γ . This means that at this order of approximation the matching point may be chosen arbitrarily as long as it is in the range $2M \ll r_m^* \ll \infty$.

Note that for $e^2/8M \approx \omega$ this expression agree with our supposition (2.15). The drastic difference between the two regimes, $\omega < e^2/8M$ and $e^2/8M < \omega$ shows up clearly if we consider the limit where the effective temperature of the black-hole tends to zero. If the self-energy of the particles is neglected, the rate of particles creation goes then to zero. In our system the rate goes also to zero if $e^2/8M < \omega$. But in the range $\mu < \omega < e^2/8M$ the rate of particle creation tends to $-\Gamma/2\pi$. We are in the domain of black-hole superradiance of charged particles.

III. PAIR PRODUCTION NEAR A SCHWARZSCHILD BLACK-HOLE

While Hawking radiance relates closely to dynamical spacetimes with horizons, pair creation does not. One may inquire for the circumstances in which a *neutral* black-hole is involved in the spontaneous production of pairs of oppositely charged particles. Technically, pair production can take place when the conditions for the existence of a “generalized ergosphere” (a region where orbits with negative total energy exist) are fulfilled. In this case, on the level of classical particles, a Penrose process [15] can take place. On the level of waves mechanics similar conditions must be fulfilled in order for “superradiant” scattering of waves obeying the Klein-Gordon equation to occur. The corresponding phenomenon that occurs then is the so called *Klein paradox* [16]. However, all this is known *not* to happen when the black-hole is a neutral static and isolated one. But, it turns out that once the black hole is allowed to interact with an *exterior* charged test object, this may give rise to pair production. Technically, this means that the conditions for the occurrence of a Klein paradox are obeyed. Then it is possible to calculate in a W.K.B. approximation the transmission coefficient through the *Klein region* separating the positive from the negative states of the corresponding Klein-Gordon equation. This gives the probability for pair creation.

Using the Hamilton-Jacobi equation Eq. (2.6), one may derive the equation of motion of a classical particle of mass μ and charge e on the background metric (2.1):

$$\left(\frac{dr}{d\tau}\right)^2 = (\varepsilon - \mathcal{E}_+(r))(\varepsilon - \mathcal{E}_-(r)),$$

$$\mathcal{E}_{\pm}(r) = \frac{e^2 M}{2r^2} \pm \sqrt{\left(1 - \frac{2M}{r}\right) \left(\mu^2 + \frac{L^2}{r^2}\right)}. \quad (3.1)$$

Here r and τ are the radial coordinate and proper time of the particle respectively, and \mathcal{E}_+ and \mathcal{E}_- are the effective potentials for the positive and negative energy solutions, respectively. The classical bound states in the potentials \mathcal{E}_{\pm} are the classical limit of the “resonances” of a quantum field satisfying the Klein-Gordon equation (2.2) written in the given background metric. Positive energy states, $\varepsilon_+ > \mathcal{E}_+$, correspond to a positive probability density, and therefore describe particles of energy ε . Negative energy states, $\varepsilon_- < \mathcal{E}_-$ correspond to a negative probability density and therefore describe antiparticles of energy $-\varepsilon$. When there is a crossing of the ε_+ and the ε_- states, the probability density has a variable sign. We are then in a Klein paradox region.

The transmission coefficient T^2 through the potential barrier separating the positive from the negative energy states is proportional to the probability for an incident particle to create a pair of particles. The transmission coefficient can be computed using the W.K.B approximation to Eq. (2.16):

$$T^2 = \exp(-\zeta),$$

$$\zeta = 2 \int_{\alpha_2}^{\alpha_1} \frac{dr}{1 - 2M/r} \sqrt{V(r)}. \quad (3.2)$$

Here ζ may be identified as the opacity of the barrier against pair creation. α_1 and α_2 are the zeroes of the effective potential $V(r)$ defined in Eq. (2.16).

What then is the energy range for which a Klein region exists? The Klein region is defined by

$$\min \mathcal{E}_+ < \varepsilon < \max \mathcal{E}_- \quad (3.3)$$

Now, \mathcal{E}_- is a monotonic decreasing function of r . It attains its maximum on the horizon: $\max \mathcal{E}_- = \mathcal{E}_-(r = 2M) = e^2/8M$. Its asymptotic value at infinity is $-\mu$. \mathcal{E}_+ has a positively diverging derivative on the horizon where it attains the same value as \mathcal{E}_- , and an asymptotic value at infinity, μ , which it approaches with a positive slope. This obviously implies that somewhere in the intermediate region between the horizon and infinity, there is a point where \mathcal{E}_+ attains a minimum, *e.g.* a potential well. The location of this minimum may be found by calculating the roots of the equation $\partial_r \mathcal{E}_+ = 0$. There are two physical roots. One determines the location of the potential minimum while the other determines the location of the potential maximum. Now, the effect of angular momentum may be ignored, since this will only strengthen the potential barrier leading to a smaller particle creation rate. Doing so, we may ask what are the allowed values of $\gamma \equiv e^2/8M\mu$ for the existence of a Klein region. Obviously, for $\gamma = 0$ ($e = 0$) which corresponds to the usual *neutral* Schwarzschild spacetime, there is no level crossing, no Klein region, and thus no particle production. This indicates that we should actually look at the other extreme, where γ is large. However, for an electron $e^2/(G\mu) \simeq 10^{15}g$. This means that for γ to be large we must look at black-holes of mass not much bigger than $10^{15}g$: the arena of mini black-holes. Keeping this in mind, we find the minimum of the effective potential \mathcal{E}_- for large γ which finally gives the energy range for the existence of a Klein region:

$$\mu \left[\sqrt{\frac{8\gamma - 3}{8\gamma - 1}} + \frac{4\gamma}{(8\gamma - 1)^2} \right] < \varepsilon < \frac{e^2}{8M}. \quad (3.4)$$

The function in the square brackets in the leftmost side of the inequality above tends rapidly to 1 as γ is increased (for $\gamma = 1$ it is equal to 0.926; for $\gamma = 5$ it is equal to 0.987). Therefore, we conclude that in practice, the energy range for the existence of a Klein region is

$$\mu < \varepsilon < \frac{e^2}{8M}. \quad (3.5)$$

This energy range is the same as the one in which the black-hole starts to superradiate (see the end of the previous section).

It should be emphasized that unlike the case where the background carries a definite sign of electric charge (like in the Kerr-Newmann spacetime), here the background is *neutral*. Therefore, it would seem that the black-hole should absorb statistically equal amounts of particles and antiparticles. Hence, it should remain neutral. In reality, a screening effect should take place. Since the point charge which creates the electric field should repel particles (or antiparticles) with the same sign of charge as its own, and attract their counterparts, a cloud of charge would form around it. This in turn should screen the point charge and further lower the pair creation rate. Furthermore, this charge segregation may alter the probabilities for assimilation of particles (antiparticles) by the black-hole, so that assimilation of particles with sign like the sign of the test charge would be more probable. Thus, the black-hole may become charged after all !

Unfortunately, calculating the corresponding *Debye length* of the problem is a formidable task and falls outside the scope of this work. Surely, the cloud of charge may be considered as a neutral gas consisting of charged particles which Coulomb-interact. Then, in order to solve the problem one must first solve Maxwell equations on the curved background with a source term representing the distribution of charge in the cloud. In the case of a completely ionized gas or plasma, this source term is given by a sum of Boltzmann's factors, one for each kind of particles, each multiplied by the charge carried by each kind of particles [17]. However, this is valid only if the pairs may be considered to be thermally distributed, which is not generally true in the case at hand.

IV. WHERE DOES THE ENERGY COME FROM? - A SPECULATION.

In the previous section it was shown that the system particle-black-hole loses energy in the process of pair creation. A simple question may be raised – what is the source of energy carried by the pairs? The corresponding problem of the possible sources of electromagnetic energy radiated away by an accelerated charge in flat spacetime, troubled and still troubles scientists [18] (it was named as the “Energy Balance Paradox”), and several answers were suggested. Leibovitz and Peres [19] suggested that there exists a charged plane, whose charge is equal and opposite in sign to the accelerated charge, and that it recedes with the speed of light in a direction opposite to the direction of the acceleration. The interaction between this charged plane and the accelerated charge supplies the energy carried by the radiation. Another suggestion, by Fulton and Rohrlich [20], is that the energy radiated is supplied from the self-energy of the charge. In the problem considered here there is a third energy source—the black-hole. Here, we show that on thermodynamical grounds the option that the black-hole lose mass during the process is not possible for massive black-holes, and discuss the proposition that the self-energy of the test charge is the energy source.

A. Thermodynamical arguments

First assume that the black-hole loses energy given by $-\Delta M$. This can be approximated by $-\Delta M \approx 2\varepsilon N$, where N is the number of pairs, and ε is some mean energy carried away by the pairs. In the nonrelativistic regime, $\varepsilon \approx \mu$ where 2μ is the pair rest-mass. In the relativistic regime the rest-mass of the particle can be neglected, so $\mu \ll \varepsilon$. Now, since the black-hole is static and neutral, its entropy is given simply by $S_{BH} = 4\pi M^2$. Therefore, during the process the black-hole changes its entropy by $\Delta S_{BH} = -8\pi M \Delta M$. Combining the two results, we find that

$$\Delta S_{BH} = -16\pi M \varepsilon N. \quad (4.1)$$

Now we make use of the generalized second law of thermodynamics (GSL). For the GSL to hold, the entropy carried out by the pairs must at least compensate for the entropy lost by the black-hole, namely

$$0 \leq \Delta S_{world} = \Delta S_{BH} + \Delta S_{pairs}. \quad (4.2)$$

Now, if the created particles are considered to be nonrelativistic, the entropy they carry is never far from the number of particles involved. Thus,

$$\Delta S_{pairs} \approx \eta N, \quad (4.3)$$

where η is a proportionality constant of order unity. The same is known to be true in the other extreme when the created pairs are assumed to be relativistic. For example, for black body radiation $\eta = 2\pi^4/(45\zeta(3)) \approx 3.6$, where $\zeta(z)$ is the Riemann zeta function [21]. Similarly, if the duration of the pair production process is long, so that the pairs are allowed to thermalize, they may be considered as particles obeying Fermi-statistics with vanishing chemical potential [17]. The specific entropy is then $\eta = S/N = 14\pi^4/(135\zeta(3)) \approx 8.4$. Substituting ΔS_{BH} and ΔS_{pairs} into the GSL we find

$$-16\pi\epsilon MN + \eta N \geq 0. \quad (4.4)$$

Hence

$$\mu M \leq \epsilon M \leq \frac{\eta}{16\pi} = O(1). \quad (4.5)$$

The conclusion must be that in the $1 \ll \mu M$ regime, the black-hole cannot be the dominant energy source; the black-hole is just too cold! Accordingly, the energy must come from somewhere else. A further conclusion is that the black-hole may not lose entropy during the process: the black-hole is involved in an adiabatic process making its horizon area invariant [22,23].

Actually, in the last derivation we have implicitly assumed that the pairs are produced at a large distance from the horizon. It turns out that the GSL even strongly forbids pair production by the black-hole taking place at close proximity to the horizon. There the energy of the pairs as measured locally is dominated by the electrostatic self-energy, and in fact diverges. To see this we note that if the constituents of a particle-antiparticle pair are considered to be quasistatic, then their conserved energy as measured at infinity, ϵ , amounts to energy invested in rest-mass plus electrostatic self-energy:

$$\begin{aligned} \epsilon &= (1 - 2M/r)\epsilon_{local} \\ &= (1 - 2M/r) \left(\mu + \frac{q^2 M}{2r(r - 2M)} \right), \end{aligned} \quad (4.6)$$

where r is the Schwarzschild coordinate of the particle (see Eq.(4.12) below). The first striking thing apparent from the expression for ϵ_{local} is that it diverges as the particle approach the horizon, $r \rightarrow 2M$. Therefore, if the pairs are located in the close proximity of the horizon then the dominant part of their energy lies in electrostatic self-energy. In the other extreme, when the particles are located far away from the horizon, their electrostatic self-energy is small compared with the rest-mass energy, until it vanishes at infinity. The point of transition, \tilde{r} , from electrostatic self-energy dominated region to rest-mass dominated region is set by the condition

$$\mu = \frac{q^2 M}{2\tilde{r}(\tilde{r} - 2M)}. \quad (4.7)$$

Thus $\tilde{r} = 2Mf(e^2/8M\mu)$, where $f(x) = \sqrt{x+1/4} + 1/2$. Note that $f(0) = 1$ - no electrostatic self-energy. As was assumed in the previous section, the pair production rate is exponentially small. Therefore in order to produce a non-negligible number of pairs, the exponent must be of order unity, *e.g.* we are limited by the condition $1 \lesssim e^2/M\mu$ (see Eq. (3.5)). Taking this into consideration we note that $f(1) \approx 1.62$. Hence, in our approximation, the electrostatic self-energy is the dominant part of the particles energy only in a narrow region around the horizon of width, $\tilde{r} - 2M \sim 0.62 \times 2M$. Pair production in that region is highly improbable on account of the GSL.

B. Dynamical arguments

Realizing that the black-hole may not be the major energy source, we turn now to look at the dynamics of the test charge as a second candidate. We begin by considering the motion of a test particle of mass m and charge e . Its motion, were it subject only to *gravitation and electromagnetic influences*, would be governed by the Lagrangian

$$L = -m \int \sqrt{-g_{\alpha\beta} u^\alpha u^\beta} d\tau + e \int A_\alpha u^\alpha d\tau, \quad (4.8)$$

where $x^\alpha(\tau)$ denotes the particle's trajectory, τ the proper time, and $u^\alpha = \dot{x}^\alpha = dx^\alpha/d\tau$, and A_α means the background electromagnetic 4-potential evaluated at the particle's spacetime position. Recalling that $g_{\alpha\beta} u^\alpha u^\beta = -1$, it follows from the Lagrangian that the canonical momenta are $p_\alpha = \delta\mathcal{L}/\delta u^\alpha = mg_{\alpha\beta} u^\beta + eA_\alpha$. The stationarity of the envisaged background means there is a timelike Killing vector $\xi^\alpha = \{1, 0, 0, 0\}$. The quantity

$$\varepsilon \equiv -p_\alpha \xi^\alpha = -mg_{t\beta} u^\beta - eA_t, \quad (4.9)$$

corresponds to the usual notion of energy as measured at infinity. Its first term expands to $m + \frac{1}{2}m(dx/dt)^2$ in the Newtonian limit. The second term, $-eA_t$, is thus the electric potential energy.

Varying the Lagrangian Eq. (4.8) with respect to u^β gives the equation of motion of the particle

$$m \frac{Du^\alpha}{d\tau} = eF_\beta^\alpha u^\beta, \quad (4.10)$$

where $F_{\alpha\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}$. The proper time derivative of ε , may be calculated as follows:

$$\dot{\varepsilon} = -\frac{d}{d\tau}(\xi^\alpha p_\alpha) = -\xi^\alpha \left(m \frac{Du_\alpha}{d\tau} + e \frac{DA_\alpha}{d\tau} \right) - eA_\alpha \frac{D\xi^\alpha}{d\tau}. \quad (4.11)$$

where we have used the fact that the proper time derivative of the timelike Killing vector, ξ^α , along the trajectory of the particle is always perpendicular to the trajectory, $(u_\alpha D\xi^\alpha/d\tau = 0)$.

Now, If the test charge is assumed to be quasistatic (supported by some mechanical apparatus), then its 4-velocity is given by $u^\beta \approx ((-g_{tt})^{-1/2}, 0, 0, 0)$. Substituting for A_α from Eq. (2.3) into Eq. (4.9) we find

$$\varepsilon = m(1 - 2M/r) + e^2 M/(2r^2). \quad (4.12)$$

Making use of the equation of motion (4.10), it is easy to show that $\dot{\varepsilon}$ vanishes. Even if we go beyond the quasistatic approximation and assume that the particle has a small but non-vanishing radial velocity, ε is still conserved. Does it mean that the system cannot radiate? We intend to show now how the equation of motion should be modified to account for the irreversible processes involved in pair production.

First, we define a 4-momentum rate of radiation by $\mathcal{R}u^\alpha$. In the case where the radiation is electromagnetic \mathcal{R} is given by the relativistic generalization of the Larmor formula $\mathcal{R} = (2/3)e^2 a^\alpha a_\alpha$; a^α is the acceleration. In the problem considered here, a simple relation between \mathcal{R} and the dynamics of the test particle is unknown. However, one may approximate the rate of energy loss due to pair production using methods to be discussed below.

$\mathcal{R}u^\alpha$, being a loss, should be subtracted from the right hand side of the equation of motion (4.10). One might expect that in this way we have correctly accounted for the momentum and energy loss due to pair production. Unfortunately, this equation is inconsistent with \mathcal{R} being positive definite; multiplication of the modified equation of motion by u_α , and using the normalization condition, $u^\alpha u_\alpha = -1$ yields $\mathcal{R} = 0$! A term must be missing. To supply it we write

$$\begin{aligned} m \frac{Du^\alpha}{d\tau} &= eF_\beta^\alpha u^\beta - \Gamma^\alpha, \\ \Gamma^\alpha &\equiv \mathcal{R}u^\alpha + S^\alpha. \end{aligned} \quad (4.13)$$

Γ^α here may be understood as a 'frictional force', and S^α is to be specified below.

There remains the question of what is the origin of that friction? One possible answer is to view the test particle as a *Brownian particle* interacting with a quantum field assumed to be in the vacuum state [24]. At zero temperature, even though the thermal fluctuations are absent, the quantum field still possess vacuum fluctuations. Obviously, the particle cannot keep accruing energy from the fluctuations present in the surrounding environment. Therefore, there should exist a mechanism for the particle to dissipate its energy so that it reaches equilibrium with the environment. Treating the quantum field as a classical stochastic variable it is possible to take the approach of Langevin who suggested, early this century, that the force exerted on the particle by the surrounding medium can effectively be written as a 'rapidly fluctuating' part and an 'averaged out' part which represents a frictional force experienced by the particle. The presence of the frictional force implies the existence of processes whereby the energy associated with the particle is dissipated to the degrees of freedom corresponding to the surrounding medium.

Multiplying Eq. (4.13) by u_α now yields

$$\Gamma_\alpha u^\alpha = 0 \quad \implies \quad \mathcal{R} = S^\alpha u_\alpha. \quad (4.14)$$

We have obtained a constraint over S^α . It is reasonable to assume that S^α is a function of the velocity and its derivatives. Given that the variation of the velocity is small, we may expand

$$S^\alpha = \mathcal{C}_0 u^\alpha + \mathcal{C}_1 a^\alpha + \mathcal{C}_2 \dot{a}^\alpha + \mathcal{C}_3 \ddot{a}^\alpha \dots, \quad (4.15)$$

where $\{\mathcal{C}_i\}$ are proportionality constants with corresponding dimensions of *energy* \times *time* ^{$i-1$} . The first term has to vanish since it is already included in $\mathcal{R}u^\alpha$. The second term may be

accounted for by mass renormalization by the rule, $m \rightarrow m + \mathcal{C}_1$ (see the equation of motion (4.13)). It turns out that the proper time derivative of the acceleration, \dot{a}^α , is the lowest derivative of the velocity allowed. Accordingly we set $S^\alpha = \mathcal{C}_2 \dot{a}^\alpha$.

In conjunction with the constraint (4.14), we must go beyond the static approximation. For if the particle is static then $u_0 u^0 = -1$, and $\mathcal{R} = u^0 S_0$. Thus, a straightforward calculation of $\dot{\varepsilon}$ using the modified equation of motion (4.13), gives

$$\dot{\varepsilon} = -\xi^\alpha \Gamma_\alpha = -(u_0 u^0 S_0 + S_0) = 0, \quad (4.16)$$

regardless of the properties of S^α . Accordingly, we set

$$u^\alpha = ((-g_{tt})^{-1/2} (1 + g_{rr} \delta u^2 / 2), \pm \delta u, 0, 0), \quad (4.17)$$

where the velocity correction, δu , is assumed to be *time independent* and small. Calculating \dot{a}^α to the lowest order in δu and its derivatives, and substituting into the constraint (4.14), we obtain a first order, non linear, ordinary differential equation for $\delta u(r)$, whose solution is:

$$\delta u(r) = \pm \left[\frac{2}{M \mathcal{C}_2} (-g_{tt})^2 \right]^{1/4} \left(\int_\infty^r r^2 \mathcal{R} dr \right)^{1/4}. \quad (4.18)$$

δu scales with $(1/M)^{1/4}$. Hence, for massive black-holes δu is small. Moreover, δu is proportional to the 1/4-th power of the energy dissipated along the trajectory of the particle, hence it depends on the *history* of the particle. \mathcal{R} must drop at least as fast as $1/r^4$ for δu to converge as $r \rightarrow \infty$. In fact, it was already assumed in Sec.III that \mathcal{R} is proportional to the transmission probability through the Klein region which is exponentially small. This however determines only the energy scale over which pair production may take place. It does not set the functional dependence of \mathcal{R} at the position of the test particle.

To resolve this we turn to Schwinger's approach [25], who showed that the probability for pair creation per unit time per unit volume by a constant electric field is

$$\left(\frac{qE}{\pi} \right)^2 \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-\frac{k\pi\mu^2}{qE}} \sqrt{-g}. \quad (4.19)$$

Multiplying this with some mean energy carried by the pairs gives an approximation for \mathcal{R} . Although (4.19) was originally derived by calculating the imaginary part of the effective Lagrangian for the Dirac field of rest-mass μ and charge q in a prescribed constant electrostatic field E in *flat space time*, we adapt this result to the corresponding problem in *curved spacetime* by substituting local expressions for the energy and electric field. We use the formula for the repulsive self-force as measured by an instantaneously comoving, freely falling observer, at the position of the test particle [2], to define the effective electric field E :

$$F_{self} = eE = \frac{e^2 M}{r^3}. \quad (4.20)$$

This repulsive force is peculiar to charged test particles. Since the black-hole is uncharged, this must be interpreted as arising from the test particle's electrostatic self-interaction. It

vanishes as M vanishes, indicating that the effect is induced by the black-hole's spacetime curvature.

Substituting for the effective electrostatic field in the formula for the rate of pair production, Eq. (4.19), we find that \mathcal{R} , vanishes exponentially fast as $r \rightarrow \infty$. Finally, putting everything together we find that

$$\frac{d\varepsilon}{d\tau} = -\mathcal{C}_2 u_0 u^r \frac{Da_r}{d\tau} = O(\mathcal{R}^{5/4}), \quad (4.21)$$

hence, the change in the energy of the test particle is negligibly small regardless of the sign of the velocity.

To summarize, as the particle is slowly lowered towards the black-hole (or pulled back) by the mechanical apparatus, additional work must be done against the frictional force induced by vacuum fluctuations. The extra energy invested in moving the particle is then dissipated away as pairs of massive charged particles.

V. SUMMARY AND ASSESSMENT

It was demonstrated how Hawking's radiance from an isolated neutral black-hole is modified on account of the electrostatic self-interaction of charged particles. Once the temperature of the black-hole is high enough so that the lightest massive charged particles are emitted, the thermal radiation of charged particles emitted by the black-hole is increased with respect to the thermal radiation of neutral particles with the same mass and spin. This is a direct consequence of the inclusion of the self-interaction into the analysis.

An interesting consequence of this conclusion is that an *external* electrostatic potential (as opposed to the *internal* electrostatic self-potential) can also be used to modulate Hawking radiance. Provided that the external potential is analytic in the neighborhood of the horizon, the thermal radiation is modulated by varying the magnitude of the potential. The modulation factor has the form $\exp(eA_t^{ext}(2M)/T_H)$, where the external potential, A_t^{ext} is evaluated at the horizon and T_H is the Hawking temperature (see Eq. (2.12)).

The possibility to modulate the thermal emission from the black-hole has some very interesting consequences. First, assume that one applies an external repulsive electrostatic field, opposing the gravitational pull of the black hole. Then the probability to propagate from the interior to the exterior of the horizon for a charged particle with energy below $eA_t^{ext}(2M)$, would be greater than the probability for the inverse process. If now the applied electrostatic field is attractive (acting in the direction of the gravitational pull), then it would serve as a high pass filter, suppressing the emission of charged particles with energy below $eA_t^{ext}(2M)$. This enable us to control the energy range over which the black-hole superradiate charged particles! The similarity between this effect and the phenomenon of thermionic emission is manifest.

Another issue that deserve further investigation is the calculation of the rate of pair production. It was assumed in Sec.III that the transmission coefficient, T^2 , through the potential barrier separating the positive from the negative energy states is proportional to the probability for an incident particle to create a pair of particles. The Klein region for the problem (including the effect of self-interaction) was found and shown to correspond to the energy range for black-hole superradiance of charged particles. Though T^2 may be

calculated numerically, it would be profitable if one could find an analytic approximation for T^2 , and show that the result converges to the result obtained using Schwinger's approach [26]. The problem is that, Schwinger's approach was originally formulated in *flat spacetime*, and the formulation of this approach in *curved spacetime* may very much prove to be an uphill task. So that a way to bypass these difficulties is much needed.

Finally, the problem of energy source for the pair production process was discussed. It was shown, that on thermodynamical grounds, it is not possible for a massive black-hole to lose mass during the process. This is just to say that the black-hole is too cold. More precisely, the entropy outflow from the system is too low for the generalized second law to hold. We thus turned to explore the dynamics of the test charge as a second candidate. It was speculated that vacuum fluctuations of a quantum field interacting with the test particle may be involved in the process of pair production. These vacuum fluctuation, induce random motion that the particle undergoes, and an averaged-out force that enters into the equation of motion as a friction term. This friction term is a manifestation of the dissipation mechanism by which energy is given off in the form of massively charged particles. That being the case, a relation between the friction term and the rate of energy dissipation, was found. As could be anticipated, a static system can not radiate. Accordingly, going beyond the static approximation, it was assumed that the test particle has a small (but non-negligible) radial velocity. Then, the functional dependence of the particle velocity on the rate of energy dissipation was determined. It was shown that the velocity scales inversely with the black-hole mass and proportional to the 1/4-th power of the energy dissipated along the trajectory of the particle, hence it depends on the *history* of the particle.

The picture that seems to arise is that as the particle is lowered towards the black hole, or pulled away, the mechanical apparatus supporting the particle is doing work in changing the particle's energy to the value appropriate to the new location. However, as the particle moves, it interacts with the vacuum fluctuations in the medium which have the effect of inducing frictional forces. If the particle is assumed to move in a constant velocity, then these frictional forces must be overcome by the investment of additional work on part of the mechanical support. The extra energy is then dissipated away in a process of pair production. Obviously for the establishment of this picture, further study of the relationship between vacuum fluctuations in *curved spacetime* and friction is in order.

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